Equilibria Refinement in Dynamic Voting Games

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Introduction

- Many decisions are made by many agents through voting
  - parliaments
  - boards
  - departments
  - families
- Common features
  - binary agenda
  - multiple decision makers
  - lack of monetary transfers

Modeling Voting

- Voting has been at the focus of game theory models since 50s
- Different ways to model voting:
  - Reduced-form models, with decision made by single representative agent (e.g., Downs 1957)
  - Sincere voting vs. strategic voting (Feddersen and Pesendorfer 1998)
- Simultaneous vs. sequential

Example: Herding

- 3 players and 2 outcomes: a and b
- $a \succ_i b$ for each i
- majority, simultaneous voting
- Then "all players vote for a" is a Nash equilibrium (and SPE)
    - But so is "all players vote for b"!

- Ways to refine that work:
    - Weakly undominated strategies
    - Trembling-hand perfect equilibrium
    - Sequential voting

Example: Two periods

- Same players, options, preferences
- Period 1: unanimity rule
  - if undecided, all get $-\varepsilon$ and proceed to period 2
- Period 2: majority voting

- Then "all vote for 2" is a SPE!
  - as before in period 2
  - but then in period 1, too, anticipating period 2

- Weakly undominated strategies refinement would not help
- THPE would not help either

Literature

- Legislative bargaining (Baron and Ferejohn 1989)
- Multi-player divide-a-dollar games (Duggan and Kalandrakis 2007)
- Dynamic coalition formation (Acemoglu, Egorov, and Sonin 2008)
- Political economy of mechanisms (Acemoglu, Golosov, and Tsyvinski 2008)

- Equilibria in extensive-form games: SPE (Selten 1975)
- Markov perfect equilibria (Maskin and Tirole 2001)
- Dominance solvable voting schemes (Moulin 1979)

This Paper

- Introduce a new equilibrium concept: Markov Trembling-Hand perfect equilibrium (MTHPE)
  - THPE with only Markovian trembles allowed
- Introduces a new concept of Agenda-Setting Game (ASG)
- In ASG, MTHPE
  - rules out counterintuitive equilibria
  - yields same predictions for games with sequential and simultaneous votings

Outline

- Model
- Agenda-Setting Games (ASG) and existing equilibrium concepts
- Sequentially Weakly Undominated equilibria (SWUE) and Markov Trembling-Hand Perfect equilibria (MTHPE): new equilibrium concepts
- Results

Model

- n players
- T stages, $T \in \mathbb{N} \cup \{\infty\}$
- action of individual i at t: $a_t^i \in A_t^i$
- $a^i = \left(a_1^i, \ldots, a_T^i\right)$: action profile of player i
- $a_t = \left(a_t^1, \ldots, a_t^n\right)$: events in period t
- $h_t = (a_1, \ldots, a_t) \in H_t$: history of play up to t
- $H^t = \bigcup_{s=0}^t H_s$: set of histories up to period t
- $a^{i,t} = \left(a_t^i, a_{t+1}^i, \ldots, a_T^i\right)$: continuation profile at t
- $a^{i,-t} = \left(a_1^i, a_2^i, \ldots, a_{t-1}^i\right)$: truncated profile at t
- $u^i\left(a^1, \ldots, a^n\right)$: utility of player i
- $u^i\left(a^{1,-t}, \ldots, a^{n,-t} \,\vdots\, a_t^1, \ldots, a_t^n \,\vdots\, a^{1,t+1}, \ldots, a^{n,t+1}\right)$: continuation utility

Strategies

- $\sigma^i : H^{T-1} \to \Delta\left(A^i\right)$: strategy of player i; set Σ^i
- $\sigma^{i,-t} : H^{t-1} \to \Delta\left(A^{i,-t}\right)$: t-truncated strategy of i
- $\sigma^{i,t} : H^{t-1} \setminus H^{t-2} \to \Delta\left(A^{i,t}\right)$: t-continuation strategy
- $u^i\left(\sigma^{i,t}, \sigma^{-i,t} \mid h^{t-1}\right)$: continuation payoff if $\sigma^{i,t}$ and $\sigma^{-i,t}$ are played after h^{t-1}

Agenda-Setting Game

- A game Γ in extensive form with n players if it satisfies:
  1. Game consists of a number of stages, where each stage is either:
     - "proposing stage" – only one player (perhaps Nature) has an action
     - "voting stage" – each player i has at most one decision node and two actions, *yes* and *no*
  2. At each voting stage Θ there are only two classes of isomorphic subgames, say $y(\Theta)$ and $n(\Theta)$
  3. Each player by switching from *no* to *yes* does not decrease the probability of moving to $y(\Theta)$
  4. If two nodes belong to the same information set, they belong to the same voting stage

Examples of ASG

- Legislative bargaining
  - agenda-setter is randomly chosen (Nature moves)
  - agenda-setter makes proposal (one agent moves)
  - voting
- Any game with no simultaneous moves

Regularization

- Suppose Γ is an agenda-setting game
- Γ' is its *regularization* if it is obtained from Γ by substituting every voting stage among set X of players by a sequence of $|X|$ stages with one player taking action in each. Game proceeds with $y(\Theta)$ in Γ' if and only if it would proceed with $y(\Theta)$ in Γ. All information sets in Γ' are trivial.
- Regularization of ASG is ASG
- Each ASG may have many regularizations
- Γ' is regularization of itself

Standard Definitions

- A strategy profile $\left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ is a *Nash Equilibrium* if

$$u^i\left(\hat{\sigma}^i, \hat{\sigma}^{-i}\right) \geq u^i\left(\sigma^i, \hat{\sigma}^{-i}\right) \text{ for all } \sigma^i \in \Sigma^i \text{ and for all } i = 1, \ldots, n.$$

- A strategy profile $\left(\hat{\sigma}^1, ..., \hat{\sigma}^N\right)$ is a *Subgame Perfect Nash Equilibrium* if

$$u^i\left(\hat{\sigma}^{i,t}, \hat{\sigma}^{-i,t} \mid h^{t-1}\right) \geq u^i\left(\sigma^{i,t}, \hat{\sigma}^{-i,t} \mid h^{t-1}\right) \text{ for all } h^{t-1} \in H^{t-1},$$

$$\text{for all } t, \text{ for all } \sigma^{-i} \in \Sigma^{-i} \text{ and for all } i = 1, \ldots, n.$$

Standard Definitions: Weakly Undominated Strategies

- A strategy profile $\left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ is a *Nash Equilibrium in Weakly Undominated Strategies* if

  1. (Nash) for each $i = 1, \ldots, n$, for any $\sigma^i \in \Sigma^i$,

  $$u^i\left(\hat{\sigma}^i, \hat{\sigma}^{-i}\right) \geq u^i\left(\sigma^i, \hat{\sigma}^{-i}\right)$$

  2. (Undominated) there does not exist $\tilde{\sigma}^i \in \Sigma^i$ such that

  $$u^i\left(\tilde{\sigma}^i, \sigma^{-i}\right) \geq u^i\left(\hat{\sigma}^i, \sigma^{-i}\right)$$

  for all $\sigma^{-i} \in \Sigma^{-i}$ with at least one strict inequality.

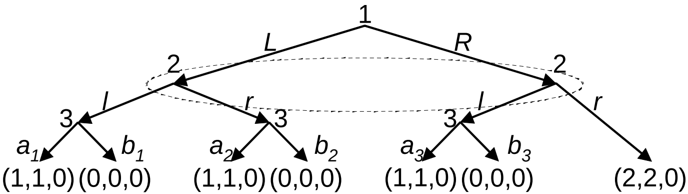
Standard Definitions: THPE

- Suppose each agent acts only once (agent-strategic form)
- Profile $\left(\hat{\sigma}^1, ..., \hat{\sigma}^n\right)$ is *THPE* if there exists a sequence of totally mixed strategy profiles $\left\{\left(\hat{\sigma}^1(m), \ldots, \hat{\sigma}^n(m)\right)\right\}_{m \in \mathbb{N}}$ such that $\left(\hat{\sigma}^1(m), \ldots, \hat{\sigma}^n(m)\right) \to \left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ as $m \to \infty$ and

$$\begin{aligned} u^i\left(\hat{\sigma}^i, \hat{\sigma}^{-i}(m)\right) &\geq u^i\left(\sigma^i, \hat{\sigma}^{-i}(m)\right) \text{ for all } \sigma^i \in \Sigma^i, \\ \text{for all } m &\in \mathbb{N} \text{ and for all } i = 1, \ldots, n. \end{aligned}$$

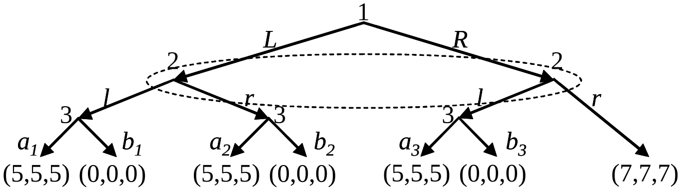
- If agents act more than once, $\left(\hat{\sigma}^1, ..., \hat{\sigma}^n\right)$ is MTHPE if it is MTHPE in the corresponding agent-strategic form game.

ASG and Weakly Undominated Strategies



- Sequential voting would yield $(R, r, (\cdot, \cdot, \cdot))$
- $(L, l, (a_1, b_2, b_3))$ only involves weakly undominated strategies
  - player 3's strategies cannot be eliminated
  - then 1's L and 2's l cannot either

ASG and Trembling-Hand Perfect Equilibrium



- $(R, r, (a_1, a_2, a_3))$ and $(L, l, (a_1, a_2, a_3))$ are both THPE
- Consider trembles
  - 1: $(1-\eta^3) L + \eta^3 R$
  - 2: $(1-\eta^3) l + \eta^3 r$
  - 3: $((1-\eta^2) a_1 + \eta^2 b_1, (1-\eta) a_2 + \eta b_2, (1-\eta) a_3 + \eta b_3)$
- Getting to (R, r) is very unlikely, so switching to R (or r) does not make sense

Sequentially Weakly Undominated Equilibrium

- Any $\left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ is h^{T+1}-SWUE
- For $t : 1 \leq t < T + 1$, $\left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ is a h^{t-1}-SWUE if
  1. (Induction) it is h^t-SWUE for any h^t that may occur after h^{t-1}
  2. (Nash) for each $i = 1, \ldots, n$, for any $\sigma^{i,t} \in \Sigma^{i,t}$,

$$u^i\left(\hat{\sigma}^{i,t}, \hat{\sigma}^{-i,t} \ \vdots\ \hat{\sigma}^{1,t+1}, \ldots, \hat{\sigma}^{N,t+1} \mid h^{t-1}\right) \geq u^i\left(\sigma^{i,t}, \hat{\sigma}^{-i,t} \ \vdots\ \hat{\sigma}^{1,t+1}, \ldots, \hat{\sigma}^{N,t+1} \mid h^{t-1}\right)$$

(continued)

- For $t : 1 \leq t < T+1$, $\left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ is a h^{t-1}-SWUE if
   3. (Undominated) there does not exist $\tilde{\sigma}^{i,t} \in \Sigma^{i,t}$ such that

$$u^i\left(\tilde{\sigma}^{i,t}, \sigma^{-i,t} \vdots \hat{\sigma}^{1,t+1}, \ldots, \hat{\sigma}^{N,t+1} \mid h^{t-1}\right) \geq u^i\left(\hat{\sigma}^{i,t}, \sigma^{-i,t} \vdots \hat{\sigma}^{1,t+1}, \ldots, \hat{\sigma}^{N,t+1} \mid h^{t-1}\right)$$

   for all $\sigma^{-i,t} \in \Sigma^{-i,t}$ with at least one strict inequality.
- Strategy profile $\left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ is a SWUE if it is h^0-SWUE

Markovian Strategy

- A continuation strategy $\sigma^{i,t}$ is Markovian if

$$\sigma^{i,t}\left(h^{t-1}\right) = \sigma^{i,t}\left(\tilde{h}^{t-1}\right)$$

for all pairs of histories $h^{t-1}, \tilde{h}^{t-1} \in H^{t-1}$ such that for any $a^{i,t}, \tilde{a}^{i,t} \in A^{i,t}$ and any $a^{-i,t} \in A^{-i,t}$

$$u^i\left(a^{i,t}, a^{-i,t} \mid h^{t-1}\right) \geq u^i\left(\tilde{a}^{i,t}, a^{-i,t} \mid h^{t-1}\right)$$

implies that

$$u^i\left(a^{i,t}, a^{-i,t} \mid \tilde{h}^{t-1}\right) \geq u^i\left(\tilde{a}^{i,t}, a^{-i,t} \mid \tilde{h}^{t-1}\right).$$

MTHPE

- Suppose each agent acts only once (agent-strategic form)
- Profile $\left(\hat{\sigma}^1, ..., \hat{\sigma}^n\right)$ is *MTHPE* if there exists a sequence of totally mixed **Markovian** strategy profiles $\left\{\left(\hat{\sigma}^1(m), \ldots, \hat{\sigma}^n(m)\right)\right\}_{m\in\mathbb{N}}$ such that $\left(\hat{\sigma}^1(m), \ldots, \hat{\sigma}^n(m)\right) \to \left(\hat{\sigma}^1, \ldots, \hat{\sigma}^n\right)$ as $m \to \infty$ and

$$\begin{aligned} u^i\left(\hat{\sigma}^i, \hat{\sigma}^{-i}(m)\right) &\geq u^i\left(\sigma^i, \hat{\sigma}^{-i}(m)\right) \text{ for all } \sigma^i \in \Sigma^i, \\ \text{for all } m &\in \mathbb{N} \text{ and for all } i = 1, \ldots, n. \end{aligned}$$

- If agents act more than once, $\left(\hat{\sigma}^1, ..., \hat{\sigma}^n\right)$ is MTHPE if it is MTHPE in the corresponding agent-strategic form game.

Existence

Theorem

Any finite or infinite extensive-form game with a finite number of stages has a MTHPE (possibly in mixed strategies).

Theorem

1. *Any (finite) agenda-setting game has a MTHPE in pure strategies.*
2. *In any finite game, a MTHPE is a SWUE.*
3. *Any finite agenda-setting game has a SWUE in pure strategies.*

SWUE = MTHPE?

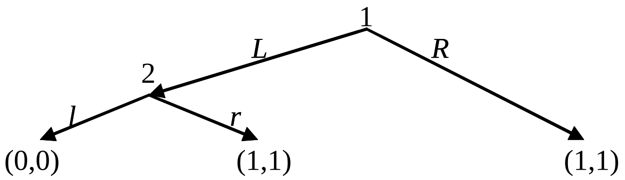


Figure:

- (R, r) is the unique MTHPE
- Two pure-strategy SWUE: (R, r) and (L, r)
- Outcomes of SWUE and MTHPE need not be payoff-equivalent
  - add dummy player 3 who gets different payoffs at (R, r) and (L, r)

Agenda-Setting games

Theorem

Suppose Γ is a finite agenda-setting game.

1. *If σ is a MTHPE of game Γ, then it is an MTHPE (and thus an SWUE and a SPE) in any regularization Γ'.*
2. *Conversely, if σ' is a MTHPE in regularization Γ', then there exists an MTHPE σ in Γ in which all players obtain the same payoffs as in σ'.*

Uniqueness of MTHPE

Theorem

Suppose agenda-setting game Γ is such that for any two terminal nodes, if one player obtains the same payoffs in these two nodes, then each player obtains the same payoffs in both nodes. Then in any two MTHPEs σ_1 and σ_2, all players obtain the same payoffs, and these payoffs are the same as those in any SPE of any regularization of Γ'.

SWUE vs. MTHPE

- MTHPE is defined for infinite games
  - divide-a-dollar game with voting
- In finite games, MTHPE is stronger
- SWUE is easier to define and compute

Conclusion

- Standard equilibria refinements fail in dynamic voting games

- This paper:
  - introduces two new equilibria concepts (SWUE and MTHPE)
  - investigate their general properties
  - establish payoff equivalence of MTHPE for agenda-setting games and their regularizations